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TRANSIENT TWO-PHASE FLOW IN LOW VELOCITY HILLY TERRAIN PIPELINES

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Abstract—For a hilly terrain pipeline operating at low flow rates of liquid and gas, the liquid tends to accumulate in the valleys while the gas tends to accumulate in the peaks. Under such conditions frictional pressure losses can be neglected and the system is totally gravity controlled. Liquid and gas is supplied at the entrance and is collected at the exit. The system can exhibit a complex transient behavior of the gas and the liquid in the pipe, even though the input flow rates of the liquid and gas are constant. The transient behavior eventually results in either a stable steady-state flow, with two-phase bubble or slug flow in the upward sections and stratified flow in the downwards sections, or complex indefinite fluctuations in pressure and fluids distributions when the liquid in the upward sections is unstable. In this work we propose a model and present the equations for predicting the behavior of such a system. Example solutions are provided for both the stable and the unstable cases.

Key Words: transient two-phase flow, hilly terrain

INTRODUCTION

For low flow rates of liquid and gas in pipelines the process is gravity dominated. Figure 1 shows a typical piping system over hilly terrain with downward inclined sections ("downcomers") followed by sections with upward inclinations ("risers"). For such a system, the liquid tends to accumulate at the bottom of each section and the gas accumulates in the peaks. Under these conditions, the pressure drop is totally governed by gravity, namely the hydrostatic pressure, and so are the liquid and gas distributions in the pipe. Although the case of a perfectly horizontal section is excluded, this is not a severe restriction since in practice pipelines are seldom perfectly horizontal.

In spite of this apparently easy to handle case, the solution for the pressure and the fluids distributions as a function of time is not trivial. In particular, one must consider the possible unstable condition that may develop in the risers as a result of the compressibility of the gas which has accumulated in the upstream peaks.

Only simple systems of this geometry have been analyzed in the past. In particular, the severe slugging phenomenon, restricted to one line, one riser and a separator of constant pressure was treated (Schmidt *et al.* 1980; Taitel 1986). More complex systems were analyzed using an elaborate numerical code, which was also claimed to simulate severe slugging (Bendiksen *et al.* 1986).

In this work we present a model for treating a general pipe system for low flow rates of gas and liquid. The model is based on a quasi-equilibrium approach that has been used by Schmidt *et al.* (1980) and has been shown to apply for a single pipeline–riser system. In this work this theory was extended to include multiple upward and downward inclinations in a way that can also handle the unstable behavior.

This model can be used as a basis for practical computer simulations of actual systems. It can also serve as a basis to check the accuracy of any elaborate numerical schemes when simulating low flow rates operations. Since the model introduced here is based on very few assumptions (related primarily only to the unstable behavior), it is expected to simulate the behavior of real systems with considerable confidence.

ANALYSIS

The pipe system considered is one that follows terrain consisting of sections of upward inclination and sections with downward slopes, as shown in figure 1. The liquid and gas enter the pipe system with a known mass flow rate of liquid, $\dot{m}_{L1}$, and gas, $\dot{m}_{G1}$. The pipe system ends with a separator that is maintained at a known pressure.

The analysis presented is valid for low velocities of liquid and gas. Frictional pressure losses can then be neglected and the liquid tends to accumulate in the valleys. We consider first the "normal" case (case 1) where the liquid accumulates in the lower sections of the pipe and the gas takes its place above the liquid, as shown in figure 1.

The upward sections of the pipe (the risers) are of lengths $s_1, s_2, \ldots, s_n$ and are inclined with angles of inclination $\gamma_1, \gamma_2, \ldots, \gamma_n$ to the horizontal. The downward inclined sections are $l_1, l_2, \ldots, l_n$ long and have angles of inclination $\beta_1, \beta_2, \ldots, \beta_n$. The liquid distribution in the pipe is given in terms of the values x_i and z_i. The liquid mass at any valley is m_{Li}, whereas the mass of the gas is designated as m_{Gi} (see figure 1 for details). Since for low flow rates the system is under hydrostatic equilibrium, the values of x_i and z_i, as well as the pressure p_i, can be calculated for given masses of liquid and gas in the pipe (m_{Li} and m_{Gi}).

A gas mass balance should satisfy

$$\frac{p_i}{RT}[(l_i - x_i)A + (s_{i-1} - z_{i-1})A] = m_{Gi}. \quad [1]$$

In this equation p_i/RT is the gas density assuming ideal gas behavior (R is the ideal gas constant and T the absolute temperature). The term in square brackets is the volume occupied by the gas.

The liquid mass balance is simply

$$\rho_L(x_i + z_i)A = m_{Li}, \quad [2]$$

where the liquid density ρ_L is assumed constant.

In [1] and [2] m_{Li} and m_{Gi} are constant for $i > 1$. m_{L1} and m_{G1}, however, increases with time owing to the input flow rate of the liquid and gas that accumulate at the first section. This accumulation leads to the unsteady behavior.

Hydrostatic pressure is satisfied by the following relation:

$$p_i = p_{i+1} + \rho_L g(z_i \sin \gamma_i - x_i \sin \beta_i). \quad [3]$$

Equations [1]–[3] consist of three equations for the three variables x_i, z_i and p_i. The solution, however, is not quite straightforward since the equations include the variables p_{i+1} and z_{i-1}. The solution is obtained by solving the resulting n equations simultaneously. Unfortunately, the equations are nonlinear and cannot be solved by a simple matrix inversion.

The solution was obtained using the following iterative procedure. The system of equations [2] and [3] were solved for x_i and z_i using initial assumed pressure p_i. The p_i values were then calculated using [1] and the values compared with the assumed values. The iteration is continued until successive values of all the p_i values are within the desired degree of accuracy. To accelerate and insure convergence, the values of x_i, z_i and p_i at the $k + 1$th iteration were taken as the means of the "old" values in the ks iteration and the new values calculated with [1]–[3]. One may attempt to use [1] and [2] to solve explicitly for x_i and z_i and use [3] to calculate p_i. This scheme, however, is highly unstable and cannot be used.

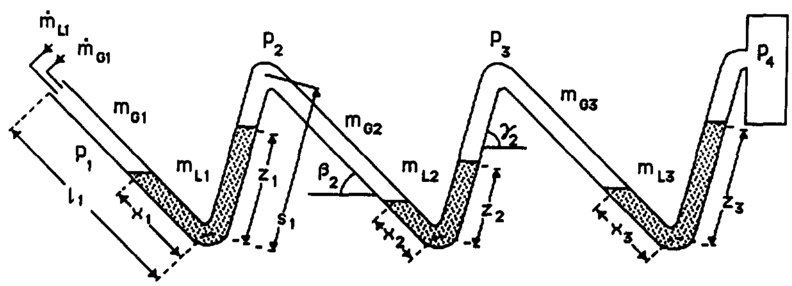

Figure 1. Pipe system—case 1.

The equations are solved for any time provided the mass of liquid in the valleys and the gas in the peaks is known. With time, however, these quantities may change. For the "normal" case described in figure 1, only the mass of the liquid and the gas in the first section increases with time, due to the liquid and gas input flow rates $\dot{m}_{L1}$ and $\dot{m}_{G1}$. However, the mass of liquid and gas in all other sections ($i > 1$) remains constant.

As liquid and gas accumulate in the first section of the pipe, one of two events may happen. For a relatively low flow rate of gas, the liquid interface in the upward section will reach the top of the pipe $z_i = s_i$ first. For a relatively high flow rate of gas and low flow rate of liquid, the liquid interface in the downward section will reach the bottom of the pipe ($x_i = 0$) before the liquid interface (z_i) reaches the top of the riser. The former is probably the more common mode of operation and will be considered first. In this case (case 2), some of the liquid from section i will be supplied to section $i + 1$ so that the liquid mass in section $i + 1$ will now increase. One may also consider that the liquid in section $i + 1$ will also reach the top of the riser so that section i will be supplying liquid to section $i + 1$ and section $i + 1$ will be supplying liquid to section $i + 2$ etc.

Figure 2 describes this typical case for section i. Equations [1]–[3] are also valid for this case, with the exception that z_i is no longer an unknown since $z_i = s_i$ and is now a constant. On the other hand, the mass of the liquid in this section is no longer a known quantity and [2] is now used to calculate the liquid mass in this section. Note also, that in this case the liquid mass in section $i + 1$ is no longer a constant since the liquid supplied from section i to the next section $i + 1$ should be taken into account.

Next we consider case 3 (see figure 3), where the liquid level in the downward inclined pipe reaches the bottom of the pipe before the liquid interface in the riser reaches to top of the pipe. In this case a two-phase steady-state flow will result. The development of the steady state can be subdivided into two steps. The first step is when the gas penetrates the liquid riser. The second step starts when the void fraction in the riser reaches the maximum value of the steady state. This maximum value ϵ_{max} will be considered known. It can be calculated separately from steady-state models or empirical correlations.

The equations that should be satisfied in this case are:

$$\frac{p_i}{RT}[l_i A + (s_{i-1} - z_{i-1})A] + \frac{p_i + p_{i+1}}{2RT} A z_i \epsilon_i = m_{Gi}, \qquad [4]$$

$$\rho_L \phi_i z_i A = m_{Li} \qquad [5]$$

and

$$p_i = p_{i+1} + \rho_L \phi_i g z_i \sin \gamma_i. \qquad [6]$$

In [4] the term $(p_i + p_{i+1})/2RT$ is the average gas density in the riser. ϕ_i is the liquid holdup ($\phi_i = 1 - \epsilon_i$). For the first step of the development to a steady state, [4]–[6] are used to calculate p_i, z_i and ϕ_i (or ϵ_i). In the second step, ϵ_i equals its maximum value ϵ_{max} and [4] is used to calculate the mass of the gas in this section. Note that the mass of the gas in this section is no longer constant since, once reaching step 2, the gas from section i flows into section $i + 1$. Furthermore, this addition of gas must also be taken into account when the section $i + 1$ is considered.

Next, case (figure 4) is considered, where the liquid interface in the riser is already at its maximum value ($z_i = s_i$) and now the liquid level in the downward inclined pipe reaches the bottom of the

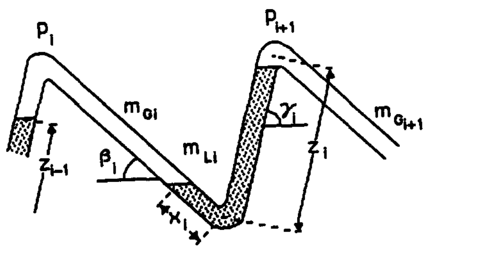

Figure 2. Case 2—liquid in the riser reaches the top of the pipe.

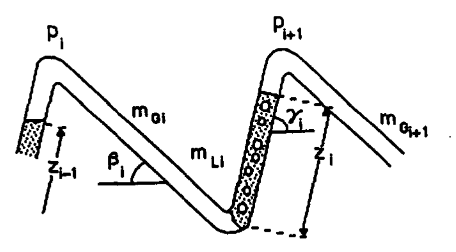

Figure 3. Case 3—liquid level reaches the bottom of the pipe.

pipe ($x_i = 0$). In this case one of two events may occur (Taitel 1986). The system may be stable and a steady two-phase flow will result. Or, the system may be unstable, resulting in a blowout of the liquid, being pushed by the upstream expanding gas. We will consider first the case where the flow is stable. This will usually occur when the volume of the gas upstream is low or the pressure is high so that the gas behaves similarly to an incompressible fluid.

As in the previous case, the development of a steady-state two-phase flow is subdivided into two steps. The first step is when the gas penetrates the liquid riser, while the second step starts when the void fraction in the riser reaches the maximum value of the steady state.

The equations that are satisfied in this case are the same as [4]–[6], with the exception that $z_i = s_i$ and the unknown z_i is replaced by the unknown m_{Li}.

Note that in this case, once reaching step 2, the flow in the riser is in the steady state. The gas and liquid that enter the bottom of the riser also exit at the top to the next $i + 1$ section.

Finally, we consider case 5, which is an unstable case. This may result only when the riser is full of liquid (or liquid with gas) and the inclined downward upstream pipe is gas filled, namely $x_i = 0$ and $z_i = s_i$. Such a situation develops from case 2 ($x_i > 0, z_i = s_i$) when x_i approaches 0, or, from case 3 ($x_i = 0, z_i < s_i$) when z_i approaches s_i. The stability analysis that provides the criterion for determining whether the flow is stable of unstable will follow later.

The unstable situation is characterized by a spontaneous expansion of the gas upstream and blowout of most of the liquid from section i to section $i + 1$. Likewise, the gas from section i channels through the liquid and is mixed with the gase of the $i + 1$ section. After this blowout process the liquid in the riser falls back and again blocks the gas passage from section i to section $i + 1$. In this process, liquid and gas from section i is spontaneously being added to section $i + 1$. At the end of this process the pressure in section $i + 1$ increases and that of section i decreases. In order to model this process we need information on the amount of fallback and also on the nature of the re-closure of the air passage as a result of the fallback. This complex process deserves a separate study. In this work it is assumed that the amount of fallback is given and the liquid that falls back to the bottom of the riser distributes equally between the riser and the line, namely $x_i = z_i$.

At the end of this spontaneous, unsteady process the situation is as shown in figure 5 and the following equations should be satisfied:

$$m_{Li} = \rho_L \phi_i s_i A(1 - \epsilon'), \quad [7]$$

$$x_i = z_i = 0.5 \frac{m_{Li}}{\rho_L A} \quad [8]$$

and

$$m_{Gi} = \frac{p_i}{RT}[(l_i - x_i)A + (s_{i-1} - z_{i-1})A], \quad [9]$$

where ϵ' is the void fraction that exists right after the blowout process in the riser. As seen, in this process the liquid and the gas are re-distributed between sections i and $i + 1$ and the section becomes a "normal" section, as in case 1.

Stability analysis

The liquid column shown in figure 4, with or without gas, can be unstable when the gas volume upstream is large. This kind of instability was analyzed by Taitel (1986) for the severe slugging

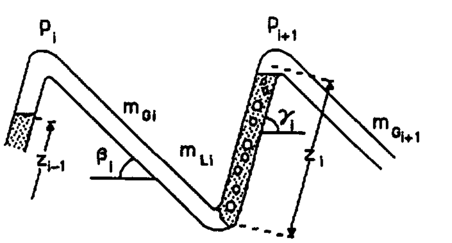

Figure 4. Case 4—steady state in the riser.

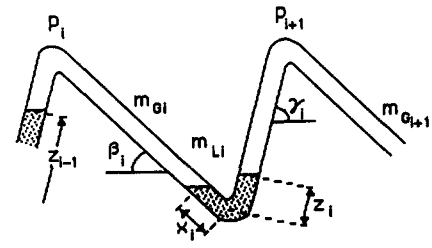

Figure 5. Case 5—after blowout and fallback.

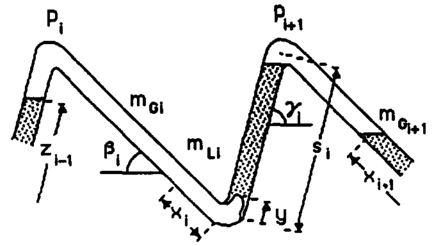

Figure 6. Stability of the liquid column in the riser.

phenomenon. The analysis, however, was confined to a single pipe and a single riser with constant separator pressure.

The analysis in this case is more complex since instability in section i may be influenced by the conditions in other sections. In this analysis we will first consider the cases when the liquid upstream as well as downstream is in the "normal" condition, while the section considered (i) has just reached the position where both $x_i = 0$ and $z_i = s_i$, as shown in figure 6.

Assume a gas bubble penetrates this column a distance y, as shown in figure 6. The net force (per unit area) acting on the liquid column in the riser is

$$\Delta F = p_i \frac{l_i + s_{i-1} - z_{i-1}}{l_i + s_{i-1} - z_{i-1} + \epsilon' y - \Delta x_{i-1}} - \rho_L g \phi_i \sin \gamma_i (s_i - y) - p_{i+1} \frac{l_{i+1} - x_{i+1}}{l_{i+1} - x_{i+1} - \epsilon' y + \Delta x_{i+1}}. \quad [10]$$

The first term on the r.h.s. is the instantaneous gas pressure at section i. It is assumed that the expansion of the gas is isothermal. p_i is the equilibrium pressure (at $y = 0$). As a result of this expansion, the force provided by the gas decreases with y and increases with Δx_{i-1} in proportion to the volume ratio as shown in [10]. ϵ' is the void fraction in the region of the bubble penetration and Δx_{i-1} is the movement of the $z_{i-1}(=x_{i-1})$ interface as a result of this spontaneous expansion. The second term corresponds to the back pressure provided by the liquid column. As can be seen, this force is proportional to $s_i - y$. As y increases this hydrostatic force decreases, which is the major cause for instability. The third term is the back pressure provided by the next $i + 1$ section. Again, it is assumed that this pressure varies with the ratio of the volumes. The volume of the gas at the $i + 1$ section decreases due to the addition of liquid (or liquid–gas mixture) that penetrates the gas volume. This additional volume is the same as the change of the volume in the is section but with an opposite sign. In addition, due to the increase of pressure in the $i + 1$ section, x_{i+1} decreases and its change is designated as Δx_{i+1}.

When $y = 0$, Δx_{i-1} and Δx_{i+1} are also zero, the system is under hydrostatic equilibrium, and $\Delta F = 0$. If, as a result of the spontaneous penetration of the gas, ΔF increases with y, then the section i is unstable and a spontaneous expansion of the gas in section i and a blowout of the liquid from section i to the next section will occur. The condition for instability is thus $\partial F/\partial y > 0$.

In order to determine the stability condition we need to express Δx_{i-1} and Δx_{i+1} in terms of y. We may assume that except for section i, all other sections are in static equilibrium during this spontaneous expansion. This assumption enables formulating the conditions for calculating Δx_{i-1} and Δx_{i+1} as follows:

$$p_{i-1} \frac{l_{i-1} - x_{i-1} + s_{i-2} - z_{i-2}}{l_{i-1} - x_{i-1} + s_{i-2} - z_{i-2} - \Delta x_{i-2} + \Delta x_{i-1}} = p_i \frac{l_i + s_{i-1} - z_{i-1}}{l_i + s_{i-1} - z_{i-1} + \epsilon' y - \Delta x_{i-1}}$$
$$+ \rho_L g[(z_{i-1} + \Delta x_{i-1}) \sin \gamma_{i-1} - (x_{i-1} \Delta x_{i-1}) \sin \beta_{i-1}] \quad [11]$$

and

$$p_{i+1} \frac{l_{i+1} - x_{i+1}}{l_{i+1} - x_{i+1} - \epsilon' y + \Delta x_{i+1}} = p_{i+2} \frac{l_{i+2} - x_{i+2} + s_{i+1} - z_{i+1}}{l_{i+2} - x_{i+2} + s_{i+1} - z_{i+1} - \Delta x_{i+1} + \Delta x_{i+2}}$$
$$+ \rho_L g [(z_{i+1} + \Delta x_{i+1}) \sin \gamma_{i+1} - (x_{i+1} - \Delta x_{i+1}) \sin \beta_{i+1}]; \quad [12]$$

[11] is an equation for Δx_{x-1} and [12] is an equation for Δx_{i+1} (as a function of y). Unfortunately, these equations cannot be solved since additional unknowns, Δx_{i+2} and Δx_{i-2}, are introduced in [11] and [12]. In principle, however, it is not difficult to continue applying hydrostatic equilibrium

upstream and downstream until we reach the constant separator pressure on the "right" and the first section on the "left" and all Δx_is could be solved simultaneously. However, this is not very convenient and as an approximation we will seek further simplifications.

First, we consider the case that the liquid in the upstream and downstream sections does not move, namely Δx_{i-1} and Δx_{i+1} are zero. This greatly simplifies the problem since the stability criterion at section i becomes independent of the other sections. This assumption yields a lower bound for the instability. As can be observed from [10], the contributions of both Δx_{i-1} and Δx_{i+1} are to decrease stability since both increase ΔF. This means that if section i is unstable using this assumption, then it is absolutely unstable and no further analysis is needed.

The second assumption, which leads to an upper bound for stability is to assume that $\Delta x_{i-2} = \Delta x_{i-1}$ and that $\Delta x_{i+1} = \Delta x_{i+2}$. This assumption is equivalent to assuming constant pressure in the upstream section $i+2$ and in the downstream section $i-1$. Since in reality the upstream pressure at $i+2$ increases, this assumption will yield the maximum movement of Δx_{i+1}. Similarly, this assumption will yield the maximum movement of the Δx_{i-1} so that the criterion based on this assumption will serve as the upper bound of stability. Namely, if it is stable according to this criterion, then it is absolutely stable. This procedure allows isolating the calculation to only the two adjacent sections.

Using the aforementioned simplifying assumption, [11] and [12] are quadratic equations for Δx_{i-1} and Δx_{i+1} that could be solved for Δx_{i-1} and Δx_{i+1} as a function of y. To a very good approximation when terms of Δx^2 and y (when compared to l) are neglected, the Δxs can be approximated by

$$\Delta x_{i-1} = \frac{\epsilon'}{1 + \dfrac{\dfrac{l_i + s_{i-1} - z_{i-1}}{p_i}}{\rho_L g(\sin\beta_{i-1} + \sin\gamma_{i-1})}} y \equiv \epsilon' K_{i-1} y \qquad [13]$$

and

$$\Delta x_{i+1} = \frac{\epsilon'}{1 + \dfrac{\dfrac{(l_{i+1} - x_{i+1})}{p_{i+1}}}{\rho_L g(\sin\beta_{i+1} + \sin\gamma_{i+1})}} y \equiv \epsilon' K_{i+1} y, \qquad [14]$$

which shows that both Δxs are directly proportional to y.

It is interesting to observe that two opposite effects of the gas at section $i+1$ on the stability of the system. The first effect is a stabilizing effect which is caused by the compression of the gas and its resistance to a blowout. This is manifested by the negative sign that precedes $\epsilon' y$ in the last term of [10]. The second effect is a destabilizing effect caused by the movement of the liquid level in the $i+1$ pipe (Δx_{i+1}). Note that when $l_{i+1} - x_{i+1} < p_{i+1}/\rho_L g(\sin\beta_{i+1} + \sin\gamma_{i+1})$, $K \approx 1$ and the two effects completely cancel each other. Likewise, the first term in [10] shows two effects, a stabilizing effect ($\epsilon' y$) and a destabilizing effect (Δx_{i-1}).

The stability criterion at $y = 0$ is obtained by differentiating [10] using [13] and [14] to yield:

$$\frac{\partial\,\Delta F}{\partial y} = -\frac{p_i \epsilon'(1 - K_{i-1})}{l_i + s_{i-1} - z_{i-1}} + \rho_L g \phi_i \sin\gamma_i - \frac{p_{i+1}\epsilon'(1 - K_{i+1})}{l_{i+1} - x_{i+1}} < 0. \qquad [15]$$

As can be seen, the first and the third terms have a stabilizing effect (note that K is always less than unity). It results due to the decrease of the gas pressure in section i and the compression of the gas in section $i+1$. When the pressure is very high the gas behavior is similar to an incompressible gas and the system is stable. The second term accounts for the instability caused by the decrease of the liquid column in this expansion process, which is the major reason for the instability.

As discussed earlier, [15] is an upper bound for stability. Namely, if it shows that the system is stable, then it is absolutely stable. Equation [15] with $K = 0$ is the criterion for stability when $\Delta x_{i-1} = \Delta x_{i+1} = 0$ which is used as the lower bound for stability. Namely, when [15] with $K = 0$ is not satisfied, the system is absolutely unstable.

We may note that usually the system is either stable or unstable by both criteria. Only seldom will the condition for stability be in the region of marginal stability such that the two limiting cases will predict ambiguous results. In this event, we leave it to the reader to determine the criterion to be used, depending on his preference of conservative design. We suggest the use of the average of the two criteria as a reasonable engineering approximation for predicting the stability criterion for this normally quite narrow region of ambiguous results.

The above-mentioned derivation is for the case where the system upstream and downstream of section i are "normal" (case 1). If the upstream as well as the downstream sections are in case 2, the analysis is almost the same and all that is needed is to use $s_{i\pm1}$ instead of $z_{i\pm1} + \Delta x_{i\pm1}$ in [11] and [12]. In [13] and [14] the terms with $\sin\gamma$ should be set equal to 0. Namely, K is calculated somewhat differently for sections in cases 1 and 2.

Finally, we note that the above analysis and the result of [15] is for conditions where upstream and downstream conditions correspond to case 1 (or case 2 when modified). When the first section is considered, Δx_{i-1} is obviously zero and so is K_{i-1}. When the last section is considered, $\Delta x_{i+1} = 0$, namely $K_{i+1} = 0$. This obviously greatly simplifies the calculations for these sections.

RESULTS AND DISCUSSION

Figures 7 and 8 show two typical runs for water and air flowing in a 2.54 cm dia pipe. The superficial velocity of the liquid is $u_{LS} = 0.1$ m/s and that of the gas under atmospheric conditions

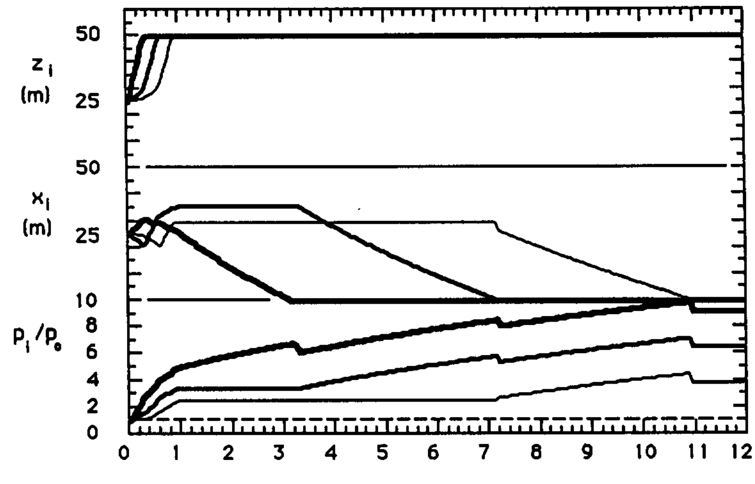

Figure 7. Stable condition, water–air. $u_{LS} = 0.1$ m/s, $u_{GSo} = 0.1$ m/s, $D = 2.54$ cm; $l_i = 50$ m, $s_i = 50$ m, $\beta_i = \gamma_i = 45°$; ——— $i = 1$, ——— $i = 2$, —— $i = 3$, - - - $i = 4$.

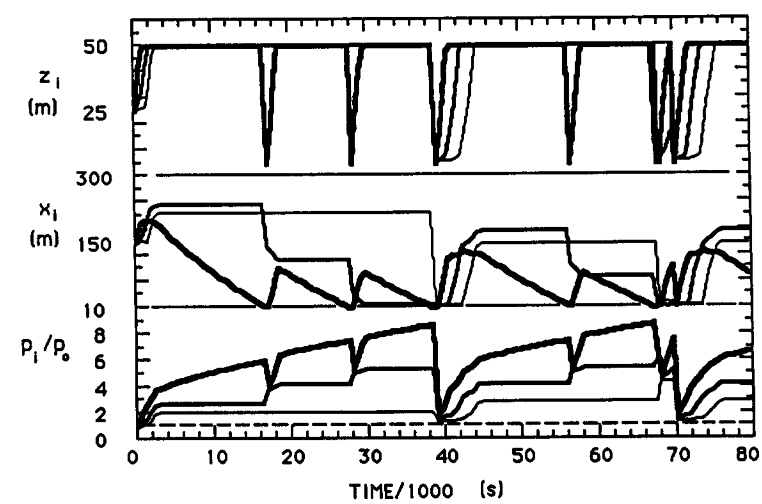

Figure 8. Unstable condition, water–air. $u_{LS} = 0.1$ m/s, $u_{GSo} = 0.1$ m/s, $D = 2.54$ cm; $l_i = 300$ m, $s_i = 50$ m, $\beta_i = 6.8°$, $\gamma = 45°$; ——— $i = 1$, ——— $i = 2$, —— $i = 3$, - - - $i = 4$.

is also $u_{GS} = 0.1$ m/s. The separator pressure, p_4, is equal to the atmospheric pressure, p_o. The basic geometry consists of three identical sections, as described in figure 1. At time $t = 0$ the pipes are half-filled with water and the pressures of all sections are atmospheric pressure. The risers lengths are 50 m and their inclination to the horizontal is 45°.

For the first run, the downcomer length is also 50 m and its inclination to the horizontal is, as for the risers, 45°. In this case the gas volume in this section is relatively small so that the system is stable and a steady state results as time progresses. the downcomers lengths for the second run are 300 m. These lengths are sufficient to cause the system to be unstable and its behavior with time is considerably different than for the first run. Obviously, a steady-state operation is never reached in this case.

Figures 7 and 8 demonstrate clearly the difference between these two runs. Figure 7 shows the case for the stable system. At time zero $z_i = x_i = 25$ m and $P_i/P_o = 1$. Liquid and gas are supplied to section 1 only. As a result, z_1, x_1 and p_1 all increase. At the same time the pressure in all the other sections increases, x_i decreases and z_i increases. Once z_1 reaches the top of the riser, the liquid in the pipe reverse its direction and x_1 decreases until it reaches the bottom of the riser. Meanwhile, the liquid in sections 2 and 3 also reaches the top of the pipe. Once $x_1 = 0$, gas penetrates the liquid column and the liquid is aerated up to its maximum value which was taken here as $\epsilon_{max} = 0.8$. A separate steady-state model shows that ϵ_{max} varies between 0.7–0.9 (depending on the specific local pressure). However, since the examples shown here are just for demonstration purposes and are not germane to the present analysis, it was decided to use an average typical value of 0.8. As a result, the pressure in section 1 decreases due to the decrease of the static head of the aerated liquid column. Next, the liquid in section 2 reaches the bottom of the pipe and similarly a steady-state flow is developed in the second riser. Finally, the liquid in the section 3 reaches the bottom of the pipe and a steady-state results for all the sections and the pressure ceases to be a function of time.

Figure 8 shows the results for an unstable system. The system is similar to the previous one with the only exception that the downcomer is six times longer. The initial process with time is similar to the previous case. However, when x_1 reaches the bottom of the pipe a blowout occurs. The liquid left in this section after blowout is taken at 20% of the liquid in the riser, and both x_1 and z_1 fall to a value of 5 m. The pressure p_1 decreases and the pressure p_2 increases as they become almost equal. Also, x_2 drops considerably due to the increase of pressure in section 2. With time the process repeats itself and a second blowout occurs at time $\approx$28,000 s. The third blowout triggers the second and the third sections to blowout too as x_i in these sections reaches the bottom of the pipe as a result of the blowout upstream. Consequently, the liquid of all three sections is blown out of the system into the separator and x_i and z_i result in a low value of 5 m in all sections. The system then starts to fill up with liquid and gas and again blowout occurs, either at the first section only or, at times, the blowout in the first section causes a blowout in sections 2 and 3 (see $t = 70{,}000$ s). This transient process continues "forever" and fluctuation of pressure and liquid and gas quantities in the pipe continue with no apparent exact repetitious cycle.

It is interesting to observe that the instability in this example started only in the first section and when the other sections were in the condition of case 2, namely the liquid level is at the top of the risers in all the sections. We also note that the determination of the stability criterion in this study was unambiguous since both bounds result in the same stable or unstable condition.

SUMMARY AND CONCLUSIONS

The present work establishes the basis for calculating pipeline system behavior under transient flow conditions for the case where the liquid and gas flow rates are low so that the frictional pressure losses can be neglected.

A solution is obtained in simple and closed algebraic relations form in which the time-dependent variables can be solved using a systematic iteration procedure.

The present model can be used for predicting pipe system behavior for both stable and unstable situations under low flow rate conditions. It can also be used as a tool for checking more elaborate transient codes and their capabilities to handle low flow rate two-phase flow conditions.

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